

On the Complexity of Hard Enumeration Problems

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Abstract. Complexity theory provides a wealth of complexity classes for analyzing the complexity of decision and counting problems. Despite the practical relevance of enumeration problems, the tools provided by complexity theory for this important class of problems are very limited. In particular, complexity classes analogous to the polynomial hierarchy and an appropriate notion of problem reduction are missing. In this work, we lay the foundations for a complexity theory of hard enumeration problems by proposing a hierarchy of complexity classes and by investigating notions of reductions for enumeration problems.

1 Introduction

While decision problems often ask for the *existence of a solution* to some problem instance, enumeration problems aim at outputting *all solutions*. In many domains, enumeration problems are thus the most natural kind of problems. Just take the database area (usually the user is interested in all answer tuples and not just in a yes/no answer) or diagnosis (where the user wants to retrieve possible explanations, and not only whether one exists) as two examples. Nevertheless, the complexity of enumeration problems is far less studied than the complexity of decision problems.

It should be noted that even simple enumeration problems may produce big output. To capture the intuition of easy to enumerate problems – despite a possibly exponential number of output values – various notions of tractable enumeration classes have been proposed in [13]. The class **DelayP** (“polynomial delay”) contains all enumeration problems where, for given instance x , (1) the time to compute the first solution, (2) the time between outputting any two solutions, and (3) the time to detect that no further solution exists, are all polynomially bounded in the size of x . The class **IncP** (“incremental polynomial time”) contains those enumeration problems where, for given instance x , the time to compute the next solution and for detecting that no further solution exists is polynomially bounded in the size of both x and of the already computed solutions. Obviously, the relationship $\text{DelayP} \subseteq \text{IncP}$ holds. In [17], the proper inclusion $\text{DelayP} \subsetneq \text{IncP}$ is mentioned. For these tractable enumeration classes, a variety of membership results exist, a few examples are [15,14,6,2,9]

There has also been work on intractable enumeration problems. Intractability of enumeration is typically proved by showing intractability of a related decision problem rather than directly proving lower bounds by relating one enumeration problem to the other. Tools for a more fine-grained analysis of intractable enumeration problems are missing to date. For instance, up to now we are not able to make a differentiated analysis of the complexity of the following typical enumeration problems:

$\Pi_k\text{SAT}^e / \Sigma_k\text{SAT}^e$
 INSTANCE: $\psi = \forall x_1 \exists x_2 \dots Q_k x_k \phi(\mathbf{x}, \mathbf{y}) / \psi = \exists x_1 \forall x_2 \dots Q_k x_k \phi(\mathbf{x}, \mathbf{y})$
 OUTPUT: All assignments for \mathbf{y} such that ψ is true.

This is in sharp contrast to decision problems, where the polynomial hierarchy is crucial for a detailed complexity analysis. As a matter of fact, it makes a big difference, if an NP-hard problem is in NP or not. Indeed, NP-complete problems have an efficient transformation into SAT and can therefore be solved by making use of powerful SAT-solvers. Similarly, problems in Σ_2^P can be solved by using ASP-solvers. Finally, also for problems on higher levels of the polynomial hierarchy, the number of quantifier alternations in the QBF-encoding matters when using QBF-solvers. For counting problems, an analogue of the polynomial hierarchy has been defined in form of the $\# \cdot \mathcal{C}$ -classes with $\mathcal{C} \in \{\text{P}, \text{coNP}, \Pi_2^P, \dots\}$ [12,19]. For enumeration problems, no such analogue has been studied.

Goal and Results. The goal of this work is to lay the foundations for a complexity theory of hard enumeration problems by defining appropriate complexity classes for intractable enumeration and a suitable notion of problem reductions. We propose to extend tractable enumeration classes by oracles. We will thus get a hierarchy of classes $\text{DelayP}^{\mathcal{C}}, \text{IncP}^{\mathcal{C}}$, where various complexity classes \mathcal{C} are used as oracles. As far as the definition of an appropriate notion of reductions is concerned, we follow the usual philosophy of reductions: if some enumeration problem can be reduced to another one, then we can use this reduction together with an enumeration algorithm for the latter problem to solve the first one. We observe that two principal kinds of reductions are used for decision problems, namely many-one reductions and Turing reductions. Similarly, we shall define a more declarative-style and a more procedural-style notion of reduction for enumeration problems. Our results are summarized below. All missing proof details can be found in the full version of this article [5].

- *Enumeration complexity classes.* In Section 3, we introduce a hierarchy of complexity classes of intractable enumeration via oracles and prove that it is strict unless the polynomial hierarchy collapses.
- *Declarative-style reductions.* In Section 4, we introduce a declarative-style notion of reductions. While they enjoy some desirable properties, we do not succeed in exhibiting complete problems under this reduction.
- *Procedural-style reductions and completeness results.* In Section 5, we introduce a procedural-style notion of reductions and show that they remedy some shortcomings of the declarative-style notion. In particular we prove completeness results. We obtain a Schaefer-like dichotomy complexity classification for the enumeration of models of generalized CNF-formulas.

2 Preliminaries

In the following, Σ denotes a finite alphabet and R denotes a polynomially bounded, binary relation $R \subseteq \Sigma^* \times \Sigma^*$, i.e., there is a polynomial p such that for all $(x, y) \in R$, $|y| \leq p(|x|)$. For every string x , $R(x) = \{y \in \Sigma^* \mid (x, y) \in R\}$. A string $y \in R(x)$ is called a *solution* for x . With a polynomially bounded, binary relation R , we can associate several natural problems:

EXIST_ R INSTANCE: $x \in \Sigma^*$ QUESTION: Exists $y \in \Sigma^*$ s.t. $(x, y) \in R$?	EXIST-ANOTHERSOL_ R / ANOTHERSOL_ R INSTANCE: $x \in \Sigma^*, Y \subseteq R(x)$ OUTPUT: Is $(R(x) \setminus Y) \neq \emptyset$? / $y \in R(x) \setminus Y$ or declare that no such y exists.
CHECK_ R INSTANCE: $(x, y) \in \Sigma^* \times \Sigma^*$ QUESTION: Is $(x, y) \in R$?	EXTSOL_ R INSTANCE: $(x, y) \in \Sigma^* \times \Sigma^*$ QUESTION: Is there some (possibly empty) $y' \in \Sigma^*$ such that $(x, yy') \in R$?

A binary relation R also gives rise to an enumeration problem, which aims at outputting the function $\text{Sol}_R : \Sigma^* \rightarrow 2^{\Sigma^*}, x \mapsto \{y \in \Sigma^* \mid (x, y) \in R\}$.

ENUM_ R INSTANCE: $x \in \Sigma^*$ OUTPUT: $R(x) = \{y \in \Sigma^* \mid (x, y) \in R\}$.

We assume the reader to be familiar with the polynomial hierarchy – the complexity classes P , NP , $\text{co}NP$ and, more generally, Δ_k^P , Σ_k^P , and Π_k^P for $k \in \{0, 1, \dots\}$. For more information of the counting hierarchy $\# \cdot \mathcal{C}$ defined via the complexity of the **CHECK_** R problem, we refer to [12].

In Section 1, we have already recalled two important tractable enumeration complexity classes, **DelayP** and **IncP** from [13]. Note that in [17,18], these classes are defined slightly differently by allowing only those **ENUM_** R problems in **DelayP** and **IncP** where the corresponding **CHECK_** R problem is in P . We adhere to the definition of tractable enumeration classes from [13].

A complexity class \mathcal{C} is *closed under a reduction* \leq_r if, for any two binary relations R_1 and R_2 we have that $R_2 \in \mathcal{C}$ and $R_1 \leq_r R_2$ imply $R_1 \in \mathcal{C}$. Furthermore, a reduction \leq_r *composes* (or: is *transitive*) if for any three binary relations R_1, R_2, R_3 , it is the case that $R_1 \leq_r R_2$ and $R_2 \leq_r R_3$ implies $R_1 \leq_r R_3$.

3 Complexity classes

In contrast to counting complexity, defining a hierarchy of enumeration problems via the **CHECK_** R problem of binary relations R is not appropriate. This can be seen by considering artificial problems obtained by padding the set of solutions of any problem with an exponential number of fake (and trivial to produce) solutions. While these fake solutions do not change the complexity of the check problem, enumerating these exponentially many fake solutions first gives an enumeration algorithm enough time to search for the non trivial ones.

Thus, we need an alternative approach for defining meaningful enumeration complexity classes. To this end, we first fix our computation model. We have already observed in the previous section that an enumeration problem may produce exponentially big output. Hence, the runtime and also the space requirements of an enumeration algorithm may be exponential in the input. Therefore, it is common (cf. [17]) to use the RAM model as a computational model, because a RAM can access parts of exponential-size data in polynomial time. We restrict ourselves here to polynomially bounded RAM machines, i.e., throughout the computation of such a machine, the size of the content of each register is polynomially bounded in the size of the input.

For enumeration, we will also make use of RAM machines with an **output-instruction**, as defined in [17]. This model can be extended further by introducing decision oracles. The input to the oracle is stored in special registers and the oracle takes consecutive non-empty registers as input. Moreover, following [1], we use a computational model that does not delete the input of an oracle call once such a call is made. For a detailed definition, refer to [17] or the appendix. It is important to note that due to the exponential runtime of an enumeration algorithm and the fact that the input to an oracle is not deleted when the oracle is executed, the input to an oracle call may eventually become exponential as well. Clearly, this can only happen if exponentially many consecutive special registers are non-empty, since we assume also each special register to be polynomially bounded.

Using this we define a collection of enumeration complexity classes via oracles:

Definition 1 (enumeration complexity classes). *Let ENUM_R be an enumeration problem, and \mathcal{C} a decision complexity class. Then we say that:*

- $\text{ENUM_R} \in \text{DelayP}^{\mathcal{C}}$ *if there is a RAM machine M with an oracle L in \mathcal{C} such that M enumerates ENUM_R with polynomial delay. The class $\text{IncP}^{\mathcal{C}}$ is defined analogously.*
- $\text{ENUM_R} \in \text{DelayP}_p^{\mathcal{C}}$ *if there is a RAM machine M with an oracle L in \mathcal{C} such that for any instance x , M enumerates $R(x)$ with polynomial delay and the size of the input to every oracle call is polynomially bounded in $|x|$.*

Note that the restriction of the oracle inputs to polynomial size only makes sense for $\text{DelayP}^{\mathcal{C}}$, where we have a discrepancy between the polynomial restriction (w.r.t. the input x) on the time between two consecutive solutions are output and the possibly exponential size (w.r.t. the input x) of oracle calls. No such discrepancy exists for $\text{IncP}^{\mathcal{C}}$, where the same polynomial upper bound w.r.t. the already computed solutions (resp. all solutions) applies both to the allowed time and to the size of the oracle calls.

We now prove several properties of these complexity classes. First, we draw a connection between the complexity of enumeration and decision problems.

It turns out that in order to study the class $\text{DelayP}_p^{\mathcal{C}}$ the EXTSOL_R problem is most relevant. Indeed, the standard enumeration algorithm [17,6], which outputs the solutions in lexicographical order, gives the following relationship.

Proposition 2. *Let R be a binary relation, $k \geq 0$, and $\mathcal{C} \in \{\Delta_k^P, \Sigma_k^P\}$. If $\text{EXTSOL}_R \in \mathcal{C}$ then $\text{ENUM}_R \in \text{DelayP}_p^{\mathcal{C}}$.*

An important class of search problems are those for which search reduces to decision, the so-called self-reducible problems. This notion can be captured by the following definition.

Definition 3 (self-reducibility). *Let \leq_T denote Turing reductions. We say that a binary relation R is self-reducible, if $\text{EXTSOL}_R \leq_T \text{EXIST}_R$,*

For self-reducible problems the above proposition can be refined as follows.

Proposition 4. *Let R be a binary relation, which is self-reducible, and $k \geq 0$. Then the following holds: $\text{EXIST}_R \in \Delta_k^P$ if and only if $\text{ENUM}_R \in \text{DelayP}_p^{\Delta_k^P}$.*

The above proposition gives a characterization of the class $\text{DelayP}_p^{\Delta_k^P}$ in terms of the complexity of decision problems in the case of self-reducible relations. Analogously, the notion of “enumeration self-reducibility” introduced by Kimelfeld and Kolaitis [14] allows a characterization of the class $\text{IncP}^{\Delta_k^P}$.

Definition 5 ([14], enumeration self-reducibility). *A binary relation R is enumeration self-reducible if $\text{ANOTHERSOL}_R \leq_T \text{EXIST-ANOTHERSOL}_R$.*

Proposition 6. *Let R be a binary relation, which is enumeration self-reducible, and $k \geq 0$. Then the following holds: $\text{EXIST-ANOTHERSOL}_R \in \Delta_k^P$ if and only if $\text{ENUM}_R \in \text{IncP}^{\Delta_k^P}$.*

We now prove that our classes provide strict hierarchies under the assumption that the polynomial hierarchy is strict.

Theorem 7. *Let $k \geq 0$. Then, unless the polynomial hierarchy collapses to the $(k+1)$ -st level,*

$$\text{DelayP}_p^{\Sigma_k^P} \subsetneq \text{DelayP}_p^{\Sigma_{k+1}^P}, \text{DelayP}^{\Sigma_k^P} \subsetneq \text{DelayP}^{\Sigma_{k+1}^P} \text{ and } \text{IncP}^{\Sigma_k^P} \subsetneq \text{IncP}^{\Sigma_{k+1}^P}$$

Proof. Let $k \geq 0$, let L be a Σ_{k+1}^P -complete problem. Define a relation $R_L = \{(x, 1) \mid x \in L\}$. It is clear that CHECK_L is Σ_{k+1}^P -complete. Moreover, the enumeration problem ENUM_L is in $\text{DelayP}_p^{\Sigma_{k+1}^P}$ (thus also in $\text{DelayP}^{\Sigma_{k+1}^P}$ and $\text{IncP}^{\Sigma_{k+1}^P}$). Assume that $\text{ENUM}_L \in \text{DelayP}_p^{\Sigma_k^P}$ (or $\text{ENUM}_L \in \text{DelayP}^{\Sigma_k^P}$ or $\text{ENUM}_L \in \text{IncP}^{\Sigma_k^P}$). Then CHECK_L can be decided in polynomial time using a Σ_k^P -oracle, meaning that $\text{CHECK}_L \in \Delta_{k+1}^P$ and thus the polynomial hierarchy collapses to the $(k+1)$ -st level.

The following proposition states that the complexity classes based on DelayP_p and DelayP , respectively, are very likely to be distinct. We refer to the definition of the exponential hierarchy in [11]. We only recall here that $\Delta_{k+1}^{\text{EXP}}$ denotes the class of decision problems decidable in exponential time with a Σ_k^P -oracle.

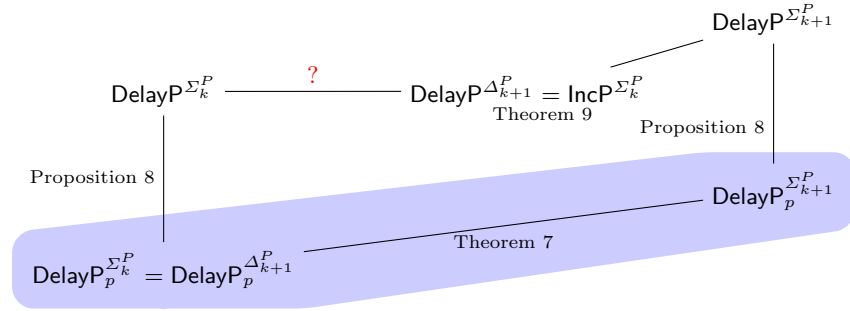


Fig. 1. Hierarchy among enumeration complexity classes for all $k \geq 1$. Solid lines without ‘?’ represent for strict inclusions under some reasonable complexity theoretic assumption. For the solid line with ‘?’, inclusion holds, but it is not clear whether it is strict. The two results (one inclusions, one equality) not discussed in this article follow immediately from the definition of the classes.

Proposition 8. *Let $k \geq 0$. If $\text{EXP} \subsetneq \Delta_{k+1}^{\text{EXP}}$, then $\text{DelayP}_p^{\Sigma_k^P} \subsetneq \text{DelayP}^{\Sigma_k^P} \not\subseteq \text{DelayP}_p^{\Sigma_{k+1}^P}$.*

I.e., the lower computational power of DelayP_p compared with DelayP or IncP cannot be compensated by equipping the lower class with a slightly more powerful oracle. While complementing this result, we now also show that in contrast, the lower computational power of DelayP compared with IncP can be compensated by equipping the lower class with a slightly more powerful oracle.

Theorem 9. *Let $k \geq 0$. Then the following holds.*

1. $\text{DelayP}_p^{\Sigma_{k+1}^P} \not\subseteq \text{DelayP}^{\Sigma_k^P}$ and $\text{DelayP}_p^{\Sigma_{k+1}^P} \not\subseteq \text{IncP}^{\Sigma_k^P}$, unless the polynomial hierarchy collapses to the $(k+1)$ -st level.
2. $\text{DelayP}^{\Delta_{k+1}^P} = \text{IncP}^{\Sigma_k^P}$.

Proof (Idea). The first claim follows from the proof of Theorem 7. For the second claim, the inclusion $\text{DelayP}^{\Delta_{k+1}^P} \subseteq \text{IncP}^{\Sigma_k^P}$ holds since the incremental delay with access to a Σ_k^P -oracle gives enough time to compute the answers of a Δ_{k+1}^P -oracle. To show that $\text{DelayP}^{\Delta_{k+1}^P} \supseteq \text{IncP}^{\Sigma_k^P}$, let $\text{ENUM_R} \in \text{IncP}^{\Sigma_k^P}$ and \mathcal{A} be a corresponding enumeration algorithm. We define a decision problem $\text{ANOTHERSOLEXT}_R^{\leq*}$ that, on an input $y_1, \dots, y_n, y', x \in \Sigma^*$, decides whether y' is the prefix of the $(n+1)$ -st output of $\mathcal{A}(x)$. Since \mathcal{A} witnesses the membership $\text{ENUM_R} \in \text{IncP}^{\Sigma_k^P}$, it follows that $\text{ANOTHERSOLEXT}_R^{\leq*} \in \Delta_{k+1}^P$, and using this language as an oracle, we have that $\text{ENUM_R} \in \text{DelayP}^{\Delta_{k+1}^P}$.

The relation among the enumeration complexity classes introduced in this chapter are summarized in Figure 1.

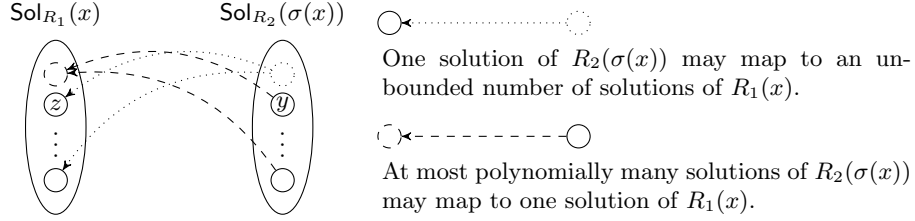


Fig. 2. Illustration of relation τ from Definition 10.

4 Declarative-style reductions

As far as we know, only a few kind of reductions between enumeration problems have been investigated so far. One such reduction is implicitly described in [7]. It establishes a bijection between sets of solutions. A different approach introduced in [3] relaxes this condition and allows non-bijective reduction functions. We go further in that direction by proposing a declarative style reduction relaxing the isomorphism requirement while preserving the enumerability.

Definition 10 (reduction \leq_e). Let $R_1, R_2 \subseteq \Sigma^*$ be binary relations. Then we define $\text{ENUM_}R_1 \leq_e \text{ENUM_}R_2$ if there exist a function $\sigma : \Sigma^* \rightarrow \Sigma^*$ computable in polynomial time and a relation $\tau \subseteq \Sigma^* \times \Sigma^* \times \Sigma^*$, s.t. for all $x \in \Sigma^*$ the following holds. For $y \in \Sigma^*$, let $\tau(x, y, -) := \{z \in \Sigma^* \mid (x, y, z) \in \tau\}$ and for $z \in \Sigma^*$, let $\tau(x, -, z) := \{y \in \Sigma^* \mid (x, y, z) \in \tau\}$. Then:

1. $\text{Sol}_{R_1}(x) = \bigcup_{y \in \text{Sol}_{R_2}(\sigma(x))} \tau(x, y, -)$;
2. $\forall y \in \text{Sol}_{R_2}(\sigma(x))$, we have $\emptyset \subsetneq \tau(x, y, -) \subseteq \text{Sol}_{R_1}(x)$ and $\tau(x, y, -)$ can be enumerated with polynomial delay in $|x|$;
3. $\forall z \in \text{Sol}_{R_1}(x)$, we have $\tau(x, -, z) \subseteq \text{Sol}_{R_2}(\sigma(x))$ and the size of $\tau(x, -, z)$ is polynomially bounded in $|x|$.

Intuitively, τ establishes a relationship between instances x , solutions $y \in \text{Sol}_{R_2}(\sigma(x))$ and solutions $z \in \text{Sol}_{R_1}(x)$. We can thus use τ to design an enumeration algorithm for $\text{Sol}_{R_1}(x)$ via an enumeration algorithm for $\text{Sol}_{R_2}(\sigma(x))$. The conditions imposed on τ have the following meaning: By condition 1, the solutions $\text{Sol}_{R_1}(x)$ can be computed by iterating through the solutions $y \in \text{Sol}_{R_2}(\sigma(x))$ and computing $\tau(x, y, -) \subseteq \text{Sol}_{R_1}(x)$. Conditions 2 and 3 make sure that the delay of enumerating $\text{Sol}_{R_1}(x)$ only differs by a polynomial from the delay of enumerating $\text{Sol}_{R_2}(\sigma(x))$: condition 2 ensures that, for every y , the set $\tau(x, y, -)$ can be enumerated with polynomial delay and that we never encounter a “useless” y (i.e., a solution $y \in \text{Sol}_{R_2}(\sigma(x))$ which is associated with no solution $z \in \text{Sol}_{R_1}(x)$). In principle, we may thus get duplicates z associated with different values of y . However, condition 3 ensures that each z can be associated with at most polynomially many values y . Figure 2 illustrates τ .

Example 11. The idea of the relation τ can also be nicely demonstrated on an \leq_e reduction from 3-COLOURABILITY^e to 4-COLOURABILITY^e (enumerating all valid 3- respectively 4-colourings of a graph). We intentionally choose this reduction since there is no bijection between the solutions of the two problems.

Recall the classical many-one reduction between these problems, which takes a graph G and defines a new graph G' by adding an auxiliary vertex v and connecting it to all the other ones. This reduction can be extended to a \leq_e reduction with the following relation τ : With every graph G in the first component of τ , we associate all valid 4-colorings (using 0, 1, 2, and 3) of G' in the third component of τ . With each of those we associate the corresponding 3-colouring of G in the second component. They are obtained from the 4-colourings by first making sure that v is coloured with 3 (by “switching” the color of v with 3) and then by simply reading off the colouring of the remaining vertices.

The reductions \leq_e have two desirable important properties, as stated next.

Proposition 12. *Let $\mathcal{C} \in \{\Sigma_k^P, \Delta_k^P \mid k \geq 0\}$. The classes $\text{DelayP}_p^{\mathcal{C}}$, $\text{DelayP}^{\mathcal{C}}$, and $\text{IncP}^{\mathcal{C}}$ are closed under \leq_e . In addition, the reductions \leq_e are transitive.*

Nevertheless their main drawback is that it is very unlikely that completeness results under \leq_e reductions can be obtained, since even the most natural problems are not complete under such a reduction.

Proposition 13. *Let $k \geq 1$. The problem $\Sigma_k\text{SAT}^e$ is not complete for $\text{DelayP}_p^{\Sigma_k^P}$ under \leq_e reductions unless the polynomial hierarchy collapses to the k^{th} level.*

5 Procedural-style reductions and completeness results

Although Turing reductions are too strong to show completeness results for classes in the polynomial hierarchy, Turing style reductions turn out to be meaningful in our case. In this section we introduce two types of reductions that are motivated by Turing reductions. Both of them are able to reduce between enumeration problems for which the reduction \leq_e seems to be too weak.

Towards this goal, we first have to define the concept of RAMs with an *oracle for enumeration problems*. The intuition behind the definition of such enumeration oracle machines is the following: For algorithms (i.e., Turing machines or RAMs in the case of enumeration) using a decision oracle for the language L , we usually have a special instruction that given an input x decides in one step whether $x \in L$, and then executes the next step of the algorithm accordingly. For an algorithm \mathcal{A} using an enumeration oracle, an input x to some ENUM- R -oracle returns in a single step (using the instruction NOO, see the definition below) a single element of $\text{Sol}_R(x)$, and then \mathcal{A} can proceed according to this output.

Definition 14 (Enumeration Oracle Machines). *Let ENUM- R be an enumeration problem. An Enumeration Oracle Machine with an enumeration oracle ENUM- R (EOM- R) is a RAM that has a sequence of new registers $O^e(0), O^e(1), \dots$*

and a new instruction **NOO** (next Oracle output). An *EOM_R* is oracle-bounded if the size of all inputs to the oracle is at most polynomial in the size of the input to the *EOM_R*.

When executing **NOO**, the machine writes – in one step – some $y_i \in \text{Sol}_R(x)$ to the accumulator A , where x is the word stored in $O^e(0), O^e(1), \dots$ and y_i is defined as follows:

Definition 15 (Next Oracle Output). *Let R be a binary relation, π_1, π_2, \dots be the run of an *EOM_R* and assume that the k^{th} instruction is **NOO**, i.e., $\pi_k = \text{NOO}$. Denote with x_i the word stored in $O^e(0), O^e(1), \dots$ at step i . Let $K = \{\pi_i \in \{\pi_1, \dots, \pi_{k-1}\} \mid \pi_i = \text{NOO} \text{ and } x_i = x_k\}$. Then the oracle output y_k in π_k is defined as an arbitrary $y_k \in \text{Sol}_R(x_k)$ s.t. y_k has not been the oracle output in any $\pi_i \in K$. If no such y_k exists, then the oracle output in π_k is undefined.*

*When executing **NOO** in step π_k , if the oracle output y_k is undefined, then the accumulator A contains some special symbol in step π_{k+1} . Otherwise in step π_{k+1} the accumulator A contains y_k .*

Observe that since an *EOM^e* is a polynomially bounded RAM and the complete oracle output is stored in the accumulator A , only such oracle calls are allowed where the size of each oracle output is guaranteed to be polynomially in the size of the input of *M^e*.

Using EOMs, we can now define another type of reductions among enumerations problems, reminiscent of classical Turing reductions. I.e., we say that one problem *ENUM_R*₁ reduces to another problem *ENUM_R*₂ if *ENUM_R*₁ can be solved by an EOM using *ENUM_R*₂ as an enumeration oracle.

Definition 16 (Reductions \leq_D, \leq_I). *Let R_1 and R_2 be binary relations.*

- *We say that $\text{ENUM}_R1 \leq_D \text{ENUM}_R2$ if there is an oracle-bounded *EOM_R*₂ that enumerates R_1 in **DelayP** and is independent of the order in which the *ENUM_R*₂ oracle enumerates its answers.*
- *We say that $\text{ENUM}_R1 \leq_I \text{ENUM}_R2$ if there is an *EOM_R*₂ that enumerates R_1 in **IncP** and is independent of the order in which the *ENUM_R*₂ oracle enumerates its answers.*

For \leq_D , we required the *EOM_R*₂ to be oracle-bounded. We would like to point out that this restriction is essential: if we drop it, then the classes **DelayP^C** are not closed under the resulting reduction. They are, however, closed under the reductions as defined above.

Proposition 17. *Let $\mathcal{C} \in \{\Sigma_k^P, \Delta_k^P \mid k \geq 0\}$. The classes **DelayP^C** and **DelayP_p^C** are closed under \leq_D . The classes **IncP^C** are closed under \leq_I .*

We note that all of these properties still hold when there is no oracle at all, i.e., for the classes **DelayP** and **IncP**.

Proposition 18. *The reductions \leq_D and \leq_I are transitive.*

Now, unlike for \leq_e , the next theorem shows that the reductions \leq_D and \leq_I induce natural complete problems for the enumeration complexity classes introduced in Section 3.

Theorem 19. *Let $k \geq 0$. Then*

1. $\Sigma_{k+1}\text{SAT}^e$ is complete for $\text{DelayP}_p^{\Sigma_{k+1}^P}$ under \leq_D reductions.
2. $\Pi_k\text{SAT}^e$ and $\Sigma_{k+1}\text{SAT}^e$ are complete for $\text{IncP}^{\Sigma_{k+1}^P}$ under \leq_I reductions.

Proof (Idea). The completeness of $\Sigma_{k+1}\text{SAT}^e$ can be shown in a similar way for both reductions. The membership is a direct consequence of Proposition 2. For hardness, we use that every input $x \in \Sigma^*$ to an L -oracle, $L \in \Sigma_{k+1}^P$, can be transformed to a quantified formula, so that we can use a corresponding $\Sigma_{k+1}\text{SAT}^e$ problem to decide whether $x \in L$. To prove the completeness of $\Pi_k\text{SAT}^e$, it suffices to show that $\Sigma_{k+1}\text{SAT}^e \leq_I \Pi_k\text{SAT}^e$. The idea is to make the variables of the additional existential quantifier part of the free variables of the formula which is an input to the $\Pi_k\text{SAT}^e$ -oracle, and repeatedly alter this formula to avoid duplicates.

Observe that, under different reductions, $\Sigma_k\text{SAT}^e$ is complete for both, $\text{IncP}^{\Sigma_k^P}$ and for the presumably smaller class $\text{DelayP}_p^{\Sigma_k^P}$. This provides additional evidence that the two reductions nicely capture $\text{IncP}^{\Sigma_k^P}$ and $\text{DelayP}_p^{\Sigma_k^P}$, respectively.

From Theorem 19 it follows as a special case that $\text{IncP}^{\Sigma_0^P}$ and $\text{IncP}^{\Sigma_1^P}$ are equivalent under \leq_I reductions: Clearly, $\Sigma_0\text{SAT}^e = \Pi_0\text{SAT}^e$, since in both cases the formulas are quantifier free and one asks for all satisfying truth assignments. Now by the theorem we know that both, $\Sigma_1\text{SAT}^e$ and $\Pi_0\text{SAT}^e$, and thus also $\Sigma_0\text{SAT}^e$, are complete for $\text{IncP}^{\Sigma_1^P}$. As a result we have that the enumeration variant of the traditional SAT problem is IncP^{NP} -complete.

For self-reducible relations we have the following general completeness result.

Theorem 20. *Let R be a binary relation and $k \geq 1$ such that EXIST_R is Σ_k^P -complete.*

- ENUM_R is $\text{DelayP}_p^{\Sigma_k^P}$ -hard under \leq_D reductions.
- ENUM_R is $\text{IncP}^{\Sigma_k^P}$ -hard under \leq_I reductions.
- If R is self-reducible, then ENUM_R is $\text{DelayP}_p^{\Sigma_k^P}$ -complete under \leq_D reductions and $\text{IncP}^{\Sigma_k^P}$ -complete under \leq_I reductions.

Proof (Idea). Let $\text{ENUM}_R' \in \text{DelayP}_p^L$ for some $L \in \Sigma_k^P$, and assume that z is the input to an L -oracle when enumerating $\text{Sol}_{R'}(x)$ for some $x \in \Sigma^*$. As EXIST_R is Σ_k^P -complete and the enumeration is oracle-bounded, z can be transformed to an equivalent instance z' of EXIST_R in time polynomial only in $|x|$. Therefore by calling the ENUM_R -oracle once and by checking whether $\text{Sol}_R(z') = \emptyset$, one can decide whether $z \in L$. The membership $\text{ENUM}_R \in \text{DelayP}_p^{\Sigma_k^P}$ in the case of self-reducibility follows by Proposition 2.

Roughly speaking Theorem 20 says that any self-reducible enumeration problem whose corresponding decision problem is hard, is hard as well. An interesting question is whether there exist easy decision problems for which the corresponding enumeration problem is hard. We answer positively to this question in revisiting, in our framework, a classification theorem obtained for the enumeration of generalized satisfiability [4]. It is convenient to first introduce some notation.

A *logical relation* of arity k is a relation $R \subseteq \{0,1\}^k$. A *constraint*, C , is a formula $C = R(x_1, \dots, x_k)$, where R is a logical relation of arity k and the x_i 's are variables. An assignment m of truth values to the variables *satisfies* the constraint C if $(m(x_1), \dots, m(x_k)) \in R$. A *constraint language* Γ is a finite set of nontrivial logical relations. A Γ -*formula* ϕ is a conjunction of constraints using only logical relations from Γ . A Γ -formula ϕ is satisfied by an assignment $m : \text{var}(\phi) \rightarrow \{0,1\}$ if m satisfies all constraints in ϕ .

Throughout the text we refer to different types of Boolean relations following Schaefer's terminology, see [16,4]. We say that a constraint language is *Schaefer* if every relation in Γ is either Horn, dualHorn, bijunctive, or affine.

SAT(Γ)^e
 INSTANCE: ϕ a Γ -formula
 OUTPUT: all satisfying assignments of ϕ .

The following theorem gives the complexity of this problem according to Γ .

Theorem 21. *Let Γ be a finite constraint language. If Γ is Schaefer, then SAT(Γ)^e is in DelayP, otherwise it is DelayP_p^{NP}-complete under \leq_D reductions.*

Proof. The polynomial cases were studied in [4]. Let us now consider the case where Γ is not Schaefer. Membership of SAT(Γ)^e in DelayP_p^{NP} is clear. For the hardness, let us introduce T and F as the two unary constant relations $T = \{1\}$ and $F = \{0\}$. According to Schaefer's dichotomy theorem [16], deciding whether a $\Gamma \cup \{F, T\}$ -formula is satisfiable is NP-complete. Since this problem is self-reducible, according to Theorem 20, SAT($\cup\{F, T\}$)^e is DelayP_p^{NP}-complete under \leq_D reductions. From the proof given in [4] it is easy to see that if Γ is not Schaefer, then SAT($\cup\{F, T\}$)^e \leq_D SAT(Γ)^e, thus concluding the proof.

To come back to the above discussion, we point out that there exist constraint languages Γ such that the decision problem SAT(Γ) is in P, while the enumeration problem SAT(Γ)^e is DelayP_p^{NP}-complete, namely 0-valid or 1-valid constraint languages that are not Schaefer.

A rather surprising completeness result is the following.

Proposition 22. *Let CIRCUMSCRIPTION^e denote the problem of enumerating all subset minimal models of a boolean formula. Then CIRCUMSCRIPTION^e is IncP^{NP}-complete under \leq_I reductions.*

What makes this result surprising is the discrepancy from the behaviour of the counting variant of the problem: The counting variant of CIRCUMSCRIPTION^e is a prototypical $\# \cdot \text{coNP}$ -complete problem [8], and thus of the same hardness as the counting variant of $\Pi_1\text{SAT}^e$. However, for enumeration we have that

CIRCUMSCRIPTION^e shows the same complexity as $\Sigma_1\text{SAT}^e$, which is considered to be lower than that of $\Pi_1\text{SAT}^e$.

Observe that CIRCUMSCRIPTION^e is very unlikely to be self-reducible: In fact, the problem of deciding if a partial truth assignment can be extended to a subset minimal model is Σ_2^P -complete [10], while deciding the existence of a minimal model is clearly NP-complete. Thus CIRCUMSCRIPTION^e is not self-reducible unless the polynomial hierarchy collapses to the first level.

6 Conclusion

We introduced a hierarchy of enumeration complexity classes, extending the well-known tractable enumeration classes DelayP and IncP, just as the Δ_k^P -classes of the polynomial-time hierarchy extend the class P. We show that under reasonable complexity assumptions these hierarchies are strict. We introduced a type of reduction among enumeration problems under which the classes in our hierarchies are closed and which allow to exhibit complete problems. For well-studied problems like Boolean CSPs in the Schaefer framework or circumscription, we obtain completeness results for the associated enumeration problems. Up to now, lower bounds for enumeration problems were of the form “ENUM_*R* is not in DelayP (or IncP) unless $P \neq NP$ ”. Our work provides a framework which allows us to pinpoint the complexity of such problems in a better way in terms of completeness.

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A Further basic definitions

We use the definitions and notations of a RAM machine as well as RAM machine using a decision oracle introduced in [17].

Definition 23 (RAM Machines).

- A RAM machine is an infinite sequence of input registers $I(0), I(1), \dots$, of computation registers $R(0), R(1) \dots$ and two special registers A and B often called accumulators, together with a finite set of instructions, called the program, of the following type:
 1. $A \leftarrow I(i)$
 2. $A \leftarrow R(i)$
 3. $B \leftarrow A$
 4. $R(i) \leftarrow B$
 5. $A \leftarrow A + B$
 6. $A \leftarrow A - B$
 7. If $A > 0$ Goto(i)
 8. Stop

- A RAM machine M is called *polynomially bounded*, if there exists some polynomial p such that on any input $x \in \Sigma^*$, the size of the content of each register of M is bounded by $p(|x|)$ on every computational step of $M(x)$.

The semantics of a RAM machine is given by its program, which is a finite sequence of instructions. Every register of the RAM machine can store words. A word encoded on a sequence of registers $R(0), R(1), \dots$ is the word $w = w_1 w_2 \dots$, where w_i is the word stored in register $R(i)$. A RAM machine with input x starts its program with x encoded in $I(0), I(1), \dots$, and all other registers are empty. We will also assume that a RAM machine always has an instruction **Output**. The output of a RAM machine M is a sequence w_1, w_2, \dots , where w_i is the word stored in the accumulator A when an **Output**-instruction is executed.

With the instruction **Output**, we can use RAM machines for enumeration. Indeed, we say that a RAM machine M enumerates an enumeration problem ENUM_R , if for all $x \in \Sigma^*$, $M(x) = \text{Sol}_R(x)$. In the following, we often use the term enumeration algorithm when we talk about a RAM machine used for enumeration.

Definition 24 (RAM Machines with decision oracles).

Let \mathcal{C} be a complexity class and $L \in \mathcal{C}$. M is called a RAM machine with an L -oracle if it is a RAM machine with new registers $O(0), O(1), \dots$ and a new instruction **Oracle**. When calling the instruction **Oracle**, M writes to A in one step the bit $\chi_L(w)$, where χ_L is the characteristic function of L and w is the word encoded on the registers $O(0), O(1), \dots$. Similar to classical complexity theory we identify L -oracles for a \mathcal{C} -complete language L as \mathcal{C} -oracles and vice versa.

Note that in an enumeration problem, the “solutions” may be output in any order – as long as no solution is output twice. Below we define the notion of an enumeration order.

Definition 25. Let R be a binary relation and $<^* \subseteq \Sigma^* \times \Sigma^*$. Then $<^*$ is an enumeration order of ENUM_R , if for every $x \in \Sigma^*$, $<^*$ is a total order on $\text{Sol}_R(x)$.

As we usually deal with exponential runtime of algorithms in the course of an enumeration process, we will also make use of the (weak) exponential hierarchy (see [11]) in the following. It is an exponential time analogue of the polynomial hierarchy defined as follows:

Definition 26 (Exponential Hierarchy).

$$\begin{aligned} \Delta_0^{\text{EXP}} &= \Sigma_0^{\text{EXP}} = \Pi_0^{\text{EXP}} = \text{EXP} \\ \Delta_k^{\text{EXP}} &= \text{EXP}^{\Sigma_{k-1}^P}, \text{ for } k \geq 1 \\ \Sigma_k^{\text{EXP}} &= \text{NEXP}^{\Sigma_{k-1}^P}, \text{ for } k \geq 1 \\ \Pi_k^{\text{EXP}} &= \text{coNEXP}^{\Sigma_{k-1}^P}, \text{ for } k \geq 1 \end{aligned}$$

For a computable function f and a language L , we will denote the class of decision problems that can be decided in time $f(n)$ with a Turing machine that has access to an L -oracle by $\text{DTIME}^L(f(n))$. Note that we sometimes use the notion when we refer to the RAM model instead of the Turing model.

B Full proofs of Section 3

As mentioned before Proposition 2, it follows immediately from the standard (one step backtracking) enumeration algorithm (cf. [17,6]). We thus omit the proof.

Proposition 4 follows immediately from Proposition 2.

Proposition 6 just generalizes [14, Proposition 2.2] from Δ_0^P to Δ_k^P for $k \geq 0$ by using of oracles and thus follows by analogous arguments.

To show Proposition 8, we first need to prove the following:

Lemma 27. *Let R be a binary relation and $k \geq 0$. If $\text{ENUM}_R \in \text{DelayP}_p^{\Sigma_k^P}$, then $\text{CHECK}_R \in \text{EXPTIME}$.*

Proof (of Lemma 27). Let (x, y) be in instance of CHECK_R and let $k \geq 0$. Further let \mathcal{A} be an enumeration algorithm witnessing the membership $\text{ENUM}_R \in \text{DelayP}_p^{\Sigma_k^P}$. It suffices to show that \mathcal{A} runs in exponential time. Indeed, in order to check whether $(x, y) \in \mathcal{A}$, we simply enumerate all of $\text{Sol}_R(x)$ and check whether $y \in \text{Sol}_R(x)$. Let q be a polynomial such that the decision of the Σ_k^P -oracle can be computed in $\mathcal{O}(2^{q(n)})$. By the definition of incremental delay and the fact that R is a polynomial relation, there is some polynomial h such that $\text{Sol}_R(x)$ can be enumerated in $\mathcal{O}(2^{q(n)+h(n)})$, which is in exponential time.

Proof (of Proposition 8).

Assume that $\text{EXP} \subsetneq \Delta_{k+1}^{\text{EXP}}$. Then there exists some polynomial q and a language L such that $L \in \Delta_{k+1}^{\text{EXP}} \setminus \text{EXP}$ and L can be decided in time $\mathcal{O}(2^{q(n)})$ using a Σ_k^P -oracle. Define the following enumeration problem:

ENUM_ D_0
 INSTANCE: $x \in \Sigma^*$.
 OUTPUT: All $\{0, 1\}^{q(|x|)}$ and 2 if $x \in L$

First note that $\text{ENUM}_D \in \text{DelayP}_p^{\Sigma_k^P}$ by an algorithm \mathcal{A} that enumerates all $2^{q(|x|)}$ words in $\{0, 1\}^{q(|x|)}$ in $\mathcal{O}(2^{q(|x|)})$. While enumerating the trivial part of the output, \mathcal{A} also has enough time to compute whether $x \in L$, and then makes the last output ('2' or nothing) accordingly. Next assume that $\text{ENUM}_D \in \text{DelayP}_p^{\Sigma_k^P}$ (or $\text{DelayP}_p^{\Sigma_{k+1}^P}$). Then by Lemma 27 $\text{CHECK}_D \in \text{EXP}$. Therefore we can check for all $x \in \Sigma^*$ whether $(x, 2) \in D_0$, which is equivalent to $x \in L$. Thus we can decide L in exponential time, a contradiction.

Proof (of Theorem 9). Let $k \geq 0$. We start the proof by showing that $\text{DelayP}^{\Delta_{k+1}^P} \supseteq \text{IncP}^{\Sigma_k^P}$. So let $\text{ENUM_R} \in \text{IncP}^{\Sigma_k^P}$ with corresponding binary relation R . Fix an incremental delay algorithm \mathcal{A} which uses a Σ_k^P oracle witnessing the membership $\text{ENUM_R} \in \text{IncP}^{\Sigma_k^P}$, and let $<^*$ be the enumeration order induced by algorithm \mathcal{A} . We define the following decision problem:

ANOTHERSOLEXT $_R^{\leq^*}$
INSTANCE: $y_1, \dots, y_n, y', x \in \Sigma^*$
QUESTION: Is y' a prefix of y_{n+1} , where y_{n+1} is the $(n+1)$ -th element in $\text{Sol}_R(x)$ w.r.t. $<^*$?

We first note that $\text{ANOTHERSOLEXT}_R^{\leq^*} \in \Delta_{k+1}^P$. Indeed, assume that we have given an instance $y_1, \dots, y_n, y', x \in \Sigma^*$. Then we can use \mathcal{A} to enumerate the first $n+1$ elements of $\text{Sol}_R(x)$ in time $\mathcal{O}(\text{poly}(|x|, |n+1|)) = \mathcal{O}(\text{poly}(|y_1| + \dots + |y_n| + |y'| + |x|))$ and then check whether y' is a prefix of y_{n+1} . As \mathcal{A} uses a Σ_k^P -oracle, this decision can be made within $\text{P}^{\Sigma_k^P} = \Delta_{k+1}^P$. The membership $\text{ENUM_R} \in \text{DelayP}^{\Delta_{k+1}^P}$ follows immediately, as we can construct a polynomial delay algorithm with an $\text{ANOTHERSOLEXT}_R^{\leq^*}$ -oracle that enumerates ENUM_R , by using a standard algorithm for enumeration.

Next we need to show that $\text{DelayP}^{\Delta_{k+1}^P} \subseteq \text{IncP}^{\Sigma_k^P}$, so let $\text{ENUM_R} \in \text{DelayP}^{\Delta_{k+1}^P}$, and let \mathcal{A} be an algorithm witnessing this membership. Moreover, let $L \in \text{P}^{\Sigma_k^P}$ be the language used for the Δ_{k+1}^P -oracle in \mathcal{A} , with a polynomial q and a language $L' \in \Sigma_k^P$ such that $L \in \text{DTIME}^{L'}(q(n))$. Let p be the polynomial for the delay of \mathcal{A} . We describe an algorithm that has an incremental delay of $p(X) \cdot Y \cdot q(p(X) \cdot Y)$ (here the indeterminate X stands for the size of the input and the indeterminate Y for the number of previously output solutions) that uses a Σ_k^P -oracle, such that \mathcal{B} enumerates ENUM_R . For this, let $x \in \Sigma^*$ and assume that we want to enumerate $\text{Sol}_R(x)$. The algorithm \mathcal{B} works as follows:

- Let y_1 be the first element output by algorithm \mathcal{A} on input x . As this output can be made by \mathcal{A} in time $p(|x|)$, at most $p(|x|)$ many calls to the L -oracle have been made with an input of size at most $p(|x|)$. Thus the answer of every oracle call can be computed in time $q(p(|x|))$ using an L' -oracle. Therefore y_1 can be output by \mathcal{B} in time $p(|x|) \cdot q(p(|x|))$ by running \mathcal{A} until the first output, and simulating the oracle calls accordingly.
- For $n \geq 2$, let y_n be the n -th element output by \mathcal{A} . As with y_1 , we can make the n -th output of \mathcal{B} by running \mathcal{A} until y_n is output and simulating the oracle calls accordingly. Indeed, \mathcal{A} takes $p(|x|) \cdot n$ steps to output y_n , with at most $p(|x|) \cdot n$ oracle calls to L with an input of size bounded by $p(|x|) \cdot n$. Thus y_n can be computed in time $p(|x|) \cdot n \cdot q(p(|x|) \cdot n)$ using an L' -oracle.

C Full proofs of Section 4

Proof (of Proposition 12). We first show that the reduction closes the enumeration classes DelayP^C , DelayP_p^C and IncP^C . Let R_1, R_2 be binary relations with

$\text{ENUM_}R_1 \leq_e \text{ENUM_}R_2$. Further let σ and τ be relations corresponding to the reduction $\text{ENUM_}R_1 \leq_e \text{ENUM_}R_2$, and assume that $\text{ENUM_}R_2 \in \text{DelayP}^C$ (the cases where $\text{ENUM_}R_2 \in \text{DelayP}_p^C$ or $\text{ENUM_}R_2 \in \text{IncP}^C$ work along the same lines). Let \mathcal{A} denote the enumeration algorithm for $\text{ENUM_}R_2$ with a polynomial delay and decision oracle \mathcal{C} , and let \mathcal{B} be the polynomial delay algorithm enumerating $\tau(x, y, -)$ for all $x, y \in \Sigma^*$. Moreover, let p be a polynomial such that for all $z \in \text{Sol}_{R_1}(x)$, we have $|\tau(x, -, z)| \leq p(|x|)$. The idea for an enumeration algorithm for $\text{ENUM_}R_1$ is to enumerate (without output) $\text{Sol}_{R_2}(\sigma(x))$ via \mathcal{A} , and for every element y that would be output by \mathcal{A} , repeatedly add $p(|x|)$ elements of $\tau(x, y, -)$ to a priority queue. Then, whenever those elements are added to the queue, an element of the queue w.r.t. some order is output. This way, one can ensure polynomial (respectively incremental) delay albeit producing an exponentially large priority queue.

To give a detailed explanation of the enumeration algorithm for $\text{ENUM_}R_1$, fix some $x \in \Sigma^*$. Denote by $\text{Newoutput}_{\mathcal{A}}(\sigma(x))$ a new output made by the enumeration algorithm \mathcal{A} when enumerating $\text{Sol}_{R_2}(\sigma(x))$, and similarly by $\text{Newoutput}_{\mathcal{B}}(y)$ a new output made by the enumeration algorithm \mathcal{B} when enumerating $\tau(x, y, -)$. Algorithm 1 gives the algorithm for enumerating $\text{Sol}_{R_1}(x)$. It is easy to see that this algorithm indeed works with a polynomial (respectively incremental) delay.

To show that the reduction is transitive, let $R_1, R_2, R_3 \subseteq \Sigma^*$ be binary relations. Further let $\text{ENUM_}R_1 \leq_e \text{ENUM_}R_2$ with corresponding polynomial p_1 and relations τ_1 and σ_1 and $\text{ENUM_}R_2 \leq_e \text{ENUM_}R_3$ with corresponding polynomial p_2 and relations τ_2 and σ_2 . Define relations σ_3 as $\sigma_3 := \sigma_2 \circ \sigma_1$ and τ_3 as

$$\tau_3 := \{(x, y, z) \in \Sigma^* \times \Sigma^* \times \Sigma^* \mid \exists \zeta \in \Sigma^* \text{ with } (x, \zeta, z) \in \tau_1 \text{ and } (\sigma_1(x), y, \zeta) \in \tau_2\}.$$

Let $x, y \in \Sigma^*$. To show that $\tau_3(x, y, -)$ can be enumerated with a polynomial delay in $|x|$, we use the same idea for a polynomial delay enumeration as we did to show that the reduction closes the enumeration classes: First a single $y' \in \text{Sol}_{R_2}(\sigma_1(x))$ is computed by $\tau_2(\sigma_1(x), y, -)$ (with a polynomial delay in $\sigma_1(|x|)$ and thus polynomial delay in $|x|$), and then at most polynomially many y'' from $\tau_1(x, y', -)$ are added to some priority queue. An element of the queue is output, and again polynomially many elements are added by the queue (possibly by first computing some new $y' \in \text{Sol}_{R_2}(\sigma_1(x))$). This way, all of $\tau_3(x, y, -)$ can be enumerated with a polynomial delay in $|x|$. Moreover, it is easy to see that for all $x, z \in \Sigma^*$ we have $|\tau_3(x, -, z)| \leq p_1(p_2(|x|))$ and that $\text{Sol}_{R_2}(x) = \bigcup_{y \in \text{Sol}_{R_3}(\sigma_3(x))} \tau_3(x, y, -)$. It follows that indeed $\text{ENUM_}R_1 \leq_e \text{ENUM_}R_3$.

For Proposition 13, we need the following lemma:

Lemma 28. *Let R_1, R_2 be binary relations and let \leq_m denote many-one reductions. Then we have: If $\text{ENUM_}R_1 \leq_e \text{ENUM_}R_2$, then $\text{EXIST_}R_1 \leq_m \text{EXIST_}R_2$.*

Proof. Let R_1, R_2 be binary relations with $\text{ENUM_}R_1 \leq_e \text{ENUM_}R_2$ and let $\sigma \in \text{FP}$ and $\tau \in \Sigma^* \times \Sigma^* \times \Sigma^*$ relations witnessing that this reduction holds. It suffices to show that for all $x \in \Sigma^*$, $x \in \text{EXIST_}R_1$ if and only if $\sigma(x) \in \text{EXIST_}R_2$.

Algorithm 1 Enumerate $\text{Sol}_{R_1}(x)$

```

1:  $i = 1$ 
2:  $\text{output\_delay} = 1$ 
3:  $\text{output\_queue} = \emptyset$ 
4:
5: while  $\mathcal{A}$  has not output all of  $\text{Sol}_{R_2}(\sigma(x))$  do
6:    $y = \text{Newoutput}_{\mathcal{A}}(\sigma(x))$ 
7:   while  $\text{output\_delay} < p(|x|)$  do
8:      $z = \text{Newoutput}_{\mathcal{B}}(y)$ 
9:     if  $z \neq \emptyset$  then
10:      Add  $z$  to  $\text{output\_queue}$ 
11:       $\text{output\_delay} = \text{output\_delay} + 1$ 
12:     else
13:       $y = \text{Newoutput}_{\mathcal{A}}(\sigma(x))$ 
14:     end if
15:     if  $y = \emptyset$  then
16:       Output (without deleting) the elements of  $\text{output\_queue}$  starting from
       the  $i$ -th element
17:        $\text{output\_delay} = p(|x|)$ 
18:     end if
19:   end while
20:   if  $y \neq \emptyset$  then
21:     Output (without deleting) the  $i$ -th element of  $\text{output\_queue}$ 
22:   end if
23:    $i = i + 1$ 
24:    $\text{output\_delay} = 1$ 
25: end while

```

So fix some x and assume that $x \in \text{EXIST_}R_1$. Then $\text{Sol}_{R_1}(x)$ is nonempty, and since $\text{Sol}_{R_1}(x) = \bigcup_{y \in \text{Sol}_{R_2}(\sigma(x))} \tau(x, y, -)$ also $\text{Sol}_{R_2}(\sigma(x))$ is nonempty, and thus $\sigma(x) \in \text{EXIST_}R_2$. For the other direction, assume that $\sigma(x) \in \text{EXIST_}R_2$. This means that there exists some $y \in \Sigma^*$ with $y \in \text{Sol}_{R_2}(\sigma(x))$. Since $\tau(x, y, -)$ is nonempty by definition and $\tau(x, y, -) \subseteq \text{Sol}_{R_1}(x)$ we are done.

Proof (of Proposition 13). Let $k \geq 1$. To show that $\Sigma_k \text{SAT}^e$ is not complete for $\text{DelayP}_p^{\Sigma_k^P}$ under \leq_e reductions, we show that there exists an enumeration problem $\text{ENUM_}R \in \text{DelayP}_p^{\Sigma_k^P}$ such that $\text{ENUM_}R \not\leq_e \Sigma_k \text{SAT}^e$ unless the polynomial hierarchy collapses to the k^{th} level. So let L be a Π_k^P -complete language and define the relation $R_L = \{(x, 1) \mid x \in L\}$. It follows immediately that $\text{EXIST_}R_L$ is also Π_k^P -complete and that $\text{ENUM_}R_L \in \text{DelayP}_p^{\Sigma_k^P}$. If $\text{ENUM_}R_L \leq_e \Sigma_k \text{SAT}^e$, then $\text{EXIST_}R_L \in \Sigma_k^P$ by Lemma 28, meaning that $\Pi_k^P \subseteq \Sigma_k^P$.

D Full proofs of Section 5

Proof (of Proposition 17). Let M be an oracle-bounded enumeration oracle machine with an enumeration oracle $\text{ENUM_}R_2$ witnessing that $\text{ENUM_}R \leq_D \text{ENUM_}R_2$. Let \mathcal{A} be the polynomial delay algorithm with access to a \mathcal{C} -oracle. We can construct a RAM machine M' that enumerates $\text{ENUM_}R_1$ with a polynomial delay using a decision oracle \mathcal{C} , by modifying the RAM machine M as follows: Every time M makes a call to an $\text{ENUM_}R_2$ oracle, we use the algorithm \mathcal{A} to retrieve what should be written to the register A . Assume that x is the input to an oracle call of the RAM machine M . Then the new RAM machine N assigns two fixed addresses a_x^0 and a_x^1 to x . Then N can simulate the algorithm \mathcal{A} on the registers $R(2^{a_x^0}), \dots, R(2^{a_x^1})$ until \mathcal{A} would output some $y \in \Sigma^*$. The RAM machine N writes y to A , and a simulation of a single oracle call is completed. Whenever x is the input for a NOO -call, N continues to simulate \mathcal{A} on those registers; this way, the enumeration of $\text{Sol}_{R_2}(x)$ does not need to start from the beginning every time x is the input of an oracle call. The proof of the closure of $\text{DelayP}_p^{\mathcal{C}}$ and $\text{IncP}^{\mathcal{C}}$ under \leq_I reductions can be done along the same lines.

Proof (of Proposition 18). This can be proven along the same lines as Proposition 17 by substituting occurrences of enumeration RAM machines with decision oracles by enumeration RAM machines with enumeration oracles.

Proof (of Theorem 19).

Let $k \geq 0$. We first show that that $\Pi_k \text{SAT}^e$ and $\Sigma_{k+1} \text{SAT}^e$ are equivalent under the \leq_I reduction. Note that $\Pi_k \text{SAT}^e \leq_I \Sigma_{k+1} \text{SAT}^e$ follows immediately from the fact that $\Pi_k \text{SAT}^e$ is a special case of $\Sigma_{k+1} \text{SAT}^e$, so it suffices to show that $\Sigma_{k+1} \text{SAT}^e \leq_I \Pi_k \text{SAT}^e$. Thus consider an instance ψ of $\Sigma_{k+1} \text{SAT}^e$ given as $\psi(x) := \exists y_0 \forall y_1 \dots Q_k y_k \phi(x, y_0, \dots, y_k)$. We can enumerate all solutions to ψ as follows: The first input to a $\Pi_k \text{SAT}^e$ oracle is

$\psi_0(x, y_0) := \forall y_1 \dots Q_k y_k \phi(x, y_0, \dots, y_k)$, with free variables x and y_0 . A single NOO instruction thus gives a solution x_0, y'_0 for ψ_0 , and x_0 can be output as a solution to ψ . The next solution can be found by calling a NOO instruction with the input $\psi_1(x, y_0) = \forall y_1 \dots Q_k y_k (\phi(x, y_0, \dots, y_k) \wedge (x_0 \neq x))$. We only need to add the clauses of $(x_0 \neq x)$ to the input registers of the oracle tape, and we can choose an encoding such that this does not alter the previous input, but extends it. The output x_1, y''_0 of the second oracle call gives the second output x_1 for the $\Sigma_{k+1}\text{SAT}^e$ problem. By repeating this method until an oracle call gives back the empty solution, we can enumerate the solutions to ψ .

Next we show that $\Sigma_k\text{SAT}^e$ is $\text{IncP}^{\Sigma_k^P}$ -complete for all $k \geq 1$ under \leq_I reductions (note that the $\text{DelayP}_p^{\Sigma_k^P}$ -completeness of $\Sigma_k\text{SAT}^e$ under \leq_D reductions can be proven along the same lines). As $\text{EXIST-}\Sigma_k\text{SAT}^e \in \Sigma_k^P$ and the relation is self-reducible, we have that $\Sigma_k\text{SAT}^e \in \text{DelayP}_p^{\Sigma_k^P}$ by Proposition 2 and thus $\Sigma_k\text{SAT}^e \in \text{IncP}^{\Sigma_k^P}$. It remains to show that for any binary relation R with $\text{ENUM-}R \in \text{IncP}^{\Sigma_k^P}$, $\text{ENUM-}R \leq_I \Sigma_k\text{SAT}^e$. Let $\mathcal{L} \in \Sigma_k^P$ and assume \mathcal{A} is the algorithm enumerating $\text{ENUM-}R$ with incremental polynomial delay and access to an oracle, which given some $x \in \Sigma^*$, tests whether $x \in \mathcal{L}$. Since $\Sigma_k\text{SAT}$ is Σ_k^P -complete, x can be transformed in polynomial time to some quantified formula ψ with $\psi = \exists y_1 \forall y_2 \dots Q_k y_k \phi(y_1, \dots, y_k)$, such that $x \in \mathcal{L}$ iff ψ is true. This transformation can be computed within the time bounds of an incremental delay. Compute an instance ψ' of $\Sigma_k\text{SAT}^e$ as $\psi'(z) = \exists y_1 \forall y_2 \dots Q_k y_k (\phi(\mathbf{y}) \wedge z)$. It follows that $x \in \mathcal{L}$ if and only if $\text{Sol}_{\Sigma_k\text{SAT}^e}(\psi) = \{1\}$. Thus whenever \mathcal{A} makes a call to a decision oracle, there is an equivalent NOO instruction to a $\Sigma_k\text{SAT}^e$ oracle, hence $\text{ENUM-}R \leq_I \Sigma_k\text{SAT}^e$ and therefore $\Sigma_k\text{SAT}^e$ is $\text{IncP}^{\Sigma_k^P}$ -hard under \leq_I reductions and thus also $\text{IncP}^{\Sigma_k^P}$ -complete.

Proof (of Theorem 20).

Let R be a relation such that $\text{EXIST-}R$ is Σ_k^P -complete for some $k \geq 1$.

- We have to prove that for any $\text{ENUM-}R' \in \text{DelayP}_p^{\Sigma_k^P}$, $\text{ENUM-}R' \leq_D \text{ENUM-}R$ ($\text{IncP}^{\Sigma_k^P}$ -hardness under \leq_I reductions can be shown along the same lines). So let R' be a binary relation such that $\text{ENUM-}R' \in \text{DelayP}_p^{\Sigma_k^P}$. By definition there is some $L \in \Sigma_k^P$ such that $\text{ENUM-}R' \in \text{DelayP}_p^L$. Moreover let \mathcal{A} be an algorithm witnessing this membership. As $\text{EXIST-}R$ is Σ_k^P -complete, we have that $L \leq_m \text{EXIST-}R$, so any input x to an L -decision oracle when enumerating $\text{ENUM-}R$ can be transformed to an instance $x' \in \text{EXIST-}R$ such that $x \in L$ iff $x' \in \text{EXIST-}R$, and this transformation can be done in polynomial time in the size of x . Moreover, since the size of the oracle input is polynomial, this reduction can be computed within the time bounds of a polynomial delay, i.e. whenever a polynomial delay algorithm with an L -oracle makes an oracle call with an input x , the same algorithm can also perform a transformation to some x' before that oracle call, without violating the polynomial delay restriction. Therefore we can enumerate $\text{ENUM-}R'$ with an oracle bounded enumeration oracle machine with $\text{ENUM-}R$ as follows:

Whenever \mathcal{A} would make a decision oracle call to L with input x , instead the machine transforms this to some $x' \in \Sigma^*$, and then makes a NOO-instruction with input x' to the ENUM- R oracle. The NOO-instruction writes a nonempty string to accumulator if and only if $x' \in \text{EXIST-}R$ and thus if and only if $x \in L$. It follows that we can simulate the decision oracle call with an enumeration oracle call.

- Membership of ENUM- R in $\text{DelayP}_p^{\Sigma_k^P}$ in this case follows immediately from Proposition 2. Since $\text{DelayP}_p^{\Sigma_k^P} \subseteq \text{IncP}^{\Sigma_k^P}$, this also shows membership in $\text{IncP}^{\Sigma_k^P}$.

Proof (of Proposition 22). To show hardness, let $\text{ENUM-}R \in \text{IncP}^{\text{NP}}$. To see that $\text{ENUM-}R \leq_I \text{CIRCUMSCRIPTION}^e$, the part proof of Theorem 19 where the hardness of $\Sigma_k\text{SAT}^e$ is shown can be adapted accordingly.

To obtain membership, we show that CIRCUMSCRIPTION^e can be enumerated in IncP^{NP} . Consider a boolean formula $\phi(x_1, \dots, x_n)$, and assume that we want to enumerate the minimal models of ϕ . We start by copying ϕ to the oracle registers. A very first minimal model x' can be achieved by a greedy algorithm using an NP oracle. Next we extend ϕ in the oracle registers to $\phi' = \phi \wedge ((x_1 < x'_1) \vee \dots \vee (x_n < x'_n))$ and again get a minimal model x'' for ϕ' using a greedy algorithm with an NP oracle. This is also a minimal model for ϕ ; repeatedly extending ϕ and then computing a minimal model via a greedy algorithm achieves the membership $\text{CIRCUMSCRIPTION}^e \in \text{DelayP}^{\text{NP}}$ and thus also $\text{CIRCUMSCRIPTION}^e \in \text{IncP}^{\text{NP}}$.